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Original article

On the well-posedness of the Cauchy problem for differential equations with distributed prehistory considering delay function perturbations

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Abstract

Theorems on the continuous dependence of the solution on perturbations of the initial data and the right-hand side of equation are proved. Under initial data we understand the collection of initial moment, of delay function and initial function. Perturbations of the right-hand side of equation are small in the integral sense.

Keywords: Well-posedness of the Cauchy problem; Differential equation with distributed delay

1. Formulation of main results

Let $I = [a, b]$ be a finite interval and $\mathbb{R}^n$ be the n-dimensional vector space of points $x = (x^1, \dots, x^n)^T$, where T denotes transposition. Suppose that $O \subset \mathbb{R}^n$ is an open set, and E_f is the set of functions $f : I \times O^2 \to \mathbb{R}^n$ satisfying the following conditions: for each fixed $(x_1, x_2) \in O^2$ the function $f(\cdot, x_1, x_2) : I \to \mathbb{R}^n$ is measurable; for each $f \in E_f$ and compact set $K \subset O$, there exist functions $m_{f,K}(t), L_{f,K}(t) \in L(I, \mathbb{R}_+)$, $\mathbb{R}_+ = [0, \infty)$, such that for almost all $t \in I$

$$|f(t, x_1, x_2)| \le m_{f,K}(t) \quad \forall (x_1, x_2) \in K^2,$$

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \le L_{f,K}(t) \sum_{i=1}^{2} |x_i - y_i|$$

$$\forall (x_1, x_2) \in K^2 \text{ and } \forall (y_1, y_2) \in K^2.$$

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We introduce a topology in E_f using the following base of neighborhoods of the origin

$$\Big\{V_{K,\delta} : K \subset O \ \text{ is a compact set and } \ \delta > 0 \ \text{ is an arbitrary number}\Big\},$$

where

$$V_{K,\delta} = \Big\{\delta f \in E_f : H(\delta f; K) \le \delta\Big\}$$

$$H(\delta f; K) = \sup\Big\{\Big|\int_{t'}^{t''} \delta f(t, x_1, x_2)dt\Big| : t', t'' \in I, x_i \in K, i = 1, 2\Big\}. \tag{1.1}$$

Let D be the set of continuous differentiable scalar functions (delay functions) $\tau(t), t \in [a, \infty)$, satisfying the conditions:

$$\tau(t) < t,\ \dot\tau(t) > 0, \quad \inf\{\tau(a) : \tau \in D\} := \hat\tau > -\infty.$$

Let $C(I_1)$ be the space of continuous functions $\varphi(t) \in \mathbb{R}^n, t \in I_1 = [\hat\tau, b]$ equipped with the norm $\|\varphi\|_{I_1} = \sup\{|\varphi(t)| : t \in I_1\}$. By $\Phi = \{\varphi \in C(I_1) : \varphi(t) \in O, t \in I_1\}$ we denote the set of initial functions.

To each element $\mu = (t_0, \tau, \varphi, f) \in A = [a, b) \times D \times \Phi \times E_f$ we assign the differential equation with distributed prehistory on the interval $[\tau(t), t]$

$$\dot x(t) = \int_{\tau(t)}^{t} f(t, x(t), x(s))ds \tag{1.2}$$

with the initial condition

$$x(t) = \varphi(t), \quad t \in [\hat\tau, t_0]. \tag{1.3}$$

Definition 1.1. Let $\mu = (t_0, \tau, \varphi, f) \in A$. A function $x(t) = x(t; \mu) \in O, t \in [\hat\tau, t_1], t_1 \in (t_0, b]$, is called a solution of Eq. (1.2) with the initial condition (1.3) or a solution corresponding to the element μ and defined on the interval $[\hat\tau, t_1]$, if it satisfies the condition (1.3), is absolutely continuous on the interval $[t_0, t_1]$ and satisfies Eq. (1.2) almost everywhere on $[t_0, t_1]$.

To formulate the main results, we introduce the following sets:

$$W(K; \alpha) = \left\{\delta f \in E_f : \exists\, m_{\delta f,K}(t), L_{\delta f,K}(t) \in L(I, R_+), \int_I [m_{\delta f,K}(t) + L_{\delta f,K}(t)]dt \le \alpha\right\},$$

where $K \subset O$ is a compact set and $\alpha > 0$ is a fixed number depending on δf;

$$B(t_{00}; \delta) = \{t_0 \in I : \ |t_0 - t_{00}| < \delta\}, \qquad V(\tau_0; \delta) = \{\tau \in D : \ \|\tau - \tau_0\|_I < \delta\},$$
$$V_1(\varphi_0; \delta) = \{\varphi \in \Phi : \ \|\varphi - \varphi_0\|_{I_1} < \delta\},$$

where $t_{00} \in [a, b)$ is a fixed point, $\tau_0 \in D$ and $\varphi_0 \in \Phi$ are fixed functions, $\delta > 0$ is a fixed number.

Theorem 1.1. *Let $x_0(t)$ be the solution corresponding to $\mu_0 = (t_{00}, \tau_0, \varphi_0, f_0) \in A$ and defined on $[\hat\tau, t_{10}], t_{10} < b$. Let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $K_0 = \varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then the following conditions hold:*

1.1. *there exist numbers $\delta_i > 0$, $i = 0, 1$, such that to each element*

$$\mu = (t_0, \tau, \varphi, f_0 + \delta f) \in V(\mu_0; K_1, \delta_0, \alpha) = B(t_{00}; \delta_0) \times V(\tau_0; \delta_0)$$
$$\times V_1(\varphi_0; \delta_0) \times \Big[f_0 + (W(K_1; \alpha) \cap V_{K_1,\delta_0})\Big]$$

corresponds solution $x(t; \mu)$ defined on the interval $[\hat\tau, t_{10} + \delta_1] \subset I_1$ and satisfying the condition $x(t; \mu) \in K_1$;

1.2. *for an arbitrary $\varepsilon > 0$, there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ such that the following inequality holds for any $\mu \in V(\mu_0; K_1, \delta_2, \alpha)$:*

$$|x(t; \mu) - x(t; \mu_0)| \le \varepsilon \quad \forall t \in [\theta, t_{10} + \delta_1], \theta = \max\{t_0, t_{00}\};$$

1.3. *for an arbitrary* $\varepsilon > 0$, *there exists a number* $\delta_3 = \delta_3(\varepsilon) \in (0, \delta_0)$ *such that the following inequality holds for any* $\mu \in V(\mu_0; K_1, \delta_3, \alpha)$*:*

$$\int_{\hat{\tau}}^{t_{10}+\delta_1} |x(t;\mu) - x(t;\mu_0)|dt \le \varepsilon.$$

Obviously, the solution $x(t;\mu_0)$ is the continuation of the solution $x_0(t)$.

In the space $E_\mu = \mathbb{R} \times D \times C(I_1) \times E_f$, we introduce the set of variations

$$\Im = \Big\{\delta\mu = (\delta t_0, \delta\tau, \delta\varphi, \delta f) \in E_\mu - \mu_0 : \ |\delta t_0| \le \beta, \|\delta\tau\|_I \le \beta,$$
$$\|\delta\varphi\|_{I_1} \le \beta, \delta f = \sum_{i=1}^{k} \lambda_i \delta f_i, |\lambda_i| \le \beta, i = \overline{1,k}\Big\},$$

where $\beta > 0$ is a fixed number and $\delta f_i \in E_f - f_0, i = \overline{1,k}$ are fixed functions.

Theorem 1.2. *Let* $x_0(t)$ *be the solution corresponding to* $\mu_0 = (t_{00}, \tau_0, \varphi_0, f_0) \in A$ *and defined on* $[\hat{\tau}, t_{10}], t_{i0} \in (a,b)$, $i = 0, 1$. *Let* $K_1 \subset O$ *be a compact set containing a certain neighborhood of the set* K_0. *Then the following conditions hold:*

1.4. *there exist numbers* $\varepsilon_1 > 0$ *and* $\delta_1 > 0$ *such that for an arbitrary* $(\varepsilon, \delta\mu) \in (0, \varepsilon_1) \times \Im$, *we have* $\mu_0 + \varepsilon\delta\mu \in A$ *and the solution* $x(t;\mu_0 + \varepsilon\delta\mu)$ *defined on the interval* $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$ *corresponds to this element. Moreover,* $x(t;\mu_0 + \varepsilon\delta\mu) \in K_1$*;*

1.5. *the following relations fulfilled:*

$$\lim_{\varepsilon\to 0}\Big[\sup\Big\{|x(t;\mu_0 + \varepsilon\delta\mu) - x(t;\mu_0)| : t \in [\theta, t_{10} + \delta_1]\Big\}\Big] = 0,$$
$$\lim_{\varepsilon\to 0}\int_{\hat{\tau}}^{t_{10}+\delta_1} |x(t;\mu_0 + \varepsilon\delta\mu) - x(t;\mu_0)|dt = 0$$

uniformly in $\delta\mu \in \Im$, *where* $\theta = \max\{t_{00}, t_{00} + \varepsilon\delta t_0\}$.

Theorem 1.2 is a simple corollary of Theorem 1.1.

Let $U_0 \subset \mathbb{R}^r$ be an open set and Ω be the set of measurable functions $u(t) \in U_0, t \in I$ satisfying the conditions: $clu(I)$ is a compact set in $\mathbb{R}^r$ and $clu(I) \subset U_0$.

To each element $\varrho = (t_0, \tau, \varphi, u) \in A_1 = [a,b) \times D \times \Phi \times \Omega$ we assign the controlled differential equation with distributed prehistory

$$\dot{x}(t) = \int_{\tau(t)}^{t} g(t, x(t), x(s), u(t))ds \tag{1.4}$$

with the initial condition (1.3). Here the function $g(t, x_1, x_2, u)$ is defined on $I \times O^2 \times U_0$ and satisfies the following conditions: for each fixed $(x_1, x_2, u) \in O^2 \times U_0$ the function $g(\cdot, x_1, x_2, u) : I \to \mathbb{R}^n$ is measurable; for each compact sets $K \subset O$ and $U \subset U_0$ there exist functions $m_{K,U}(t), L_{K,U}(t) \in L(I, R_+)$ such that for almost all $t \in I$

$$|g(t, x_1, x_2, u)| \le m_{K,U}(t) \quad \forall (x_1, x_2, u) \in K^2 \times U,$$
$$|g(t, x_1, x_2, u_1) - g(t, y_1, y_2, u_2)| \le L_{K,U}(t)\Big[\sum_{i=1}^{2}|x_i - y_i| + |u_1 - u_2|\Big]$$
$$\forall (x_1, x_2) \in K^2, \ \forall (y_1, y_2) \in K^2 \text{ and } (u_1, u_2) \in U^2.$$

Definition 1.2. Let $\varrho = (t_0, \tau, \varphi, u) \in A_1$. A function $x(t) = x(t;\varrho) \in O, t \in [\hat{\tau}, t_1], t_1 \in (t_0, b]$, is called a solution of Eq. (1.4) with the initial condition (1.3) or a solution corresponding to the element ϱ and defined on the interval $[\hat{\tau}, t_1]$, if it satisfies condition (1.3), is absolutely continuous on the interval $[t_0, t_1]$ and satisfies Eq. (1.4) almost everywhere on $[t_0, t_1]$.

Theorem 1.3. *Let $x_0(t)$ be the solution corresponding to $\varrho_0 = (t_{00}, \tau_0, \varphi_0, u_0) \in A_1$ and defined on $[\hat{\tau}, t_{10}]$, $t_{10} < b$. Let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $\varphi_0(I_1) \cup x_0([t_{00}, t_{10}])$. Then the following conditions hold:*

1.6. *there exist numbers $\delta_i > 0, i = 0, 1$ such that to each element*

$$\varrho = (t_0, \tau, \varphi, u) \in \hat{V}(\varrho_0; \delta_0) = B(t_{00}; \delta_0) \times V(\tau_0; \delta_0) \times V_1(\varphi_0; \delta_0) \times V_2(u_0; \delta_0)$$

corresponds solution $x(t; \varrho)$ defined on the interval $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$ and satisfying the condition $x(t; \varrho) \in K_1$, here $V_2(u_0; \delta_0) = \{u \in \Omega : \|u - u_0\|_I < \delta_0\}$;

1.7. *for an arbitrary $\varepsilon > 0$, there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ such that the following inequality holds for any $\varrho \in \hat{V}(\varrho_0; \delta_2)$:*

$$|x(t; \rho) - x(t; \rho_0)| \le \varepsilon \quad \forall t \in [\theta, t_{10} + \delta_1], \theta = \max\{t_0, t_{00}\};$$

1.8. *for an arbitrary $\varepsilon > 0$, there exists a number $\delta_3 = \delta_3(\varepsilon) \in (0, \delta_0)$ such that the following inequality fulfilled for any $\varrho \in \hat{V}(\varrho_0; \delta_3)$:*

$$\int_{\hat{\tau}}^{t_{10}+\delta_1} |x(t; \varrho) - x(t; \varrho_0)| dt \le \varepsilon.$$

Some comments. In Theorem 1.1 perturbations of the right-hand side of Eq. (1.2) are small in the integral sense (see (1.1)). Theorems 1.1–1.3 and their like theorems play an important role in the theory of optimal control, in proving variation formulas of solution, in the sensitivity analysis of equations [1–7]. Theorem analogous to Theorem 1.1 without perturbations of constant delay are proved in [8]. Theorems on the continuous dependence of the solution for various classes of ordinary and functional differential equations for the case in which the perturbation of the right-hand side is small in the integral sense are given in [1,5,9–13,7,14].

2. Proof of Theorem 1.1

On the continuous dependence of solution for a class of functional differential equations. To each element $\mu \in A$ we assign the functional differential equation

$$\dot{y}(t) = \int_{\tau(t)}^{t} f(t, y(t), h(t_0, \varphi, y)(s)) ds \tag{2.1}$$

with the initial condition

$$y(t_0) = \varphi(t_0), \tag{2.2}$$

where $h : I \times \Phi \times C(I) \to C(I_1)$ is the operator given by the formula

$$h(t_0, \varphi, y)(t) = \begin{cases} \varphi(t) & \text{for } t \in [\hat{\tau}, t_0), \\ y(t) & \text{for } t \in [t_0, b]. \end{cases}$$

Definition 2.1. An absolutely continuous function $y(t) = y(t; \mu) \in O, t \in [r_1, r_2] \subset I$, is called a solution of Eq. (2.1) with the initial condition (2.2) or the solution corresponding to the element $\mu \in A$ and defined on $[r_1, r_2]$, if $t_0 \in [r_1, r_2]$, $y(t_0) = \varphi(t_0)$ and satisfies Eq. (2.1) almost everywhere on the interval $[r_1, r_2]$.

Remark 2.1. Let $y(t; \mu), t \in [r_1, r_2], \mu \in A$ be the solution of Eq. (2.1) with the initial condition (2.2). Then, as is easily seen, the function

$$x(t; \mu) = h(t_0, \varphi, y(\cdot; \mu))(t), \quad t \in [\hat{\tau}, r_2]$$

is the solution of Eq. (1.2) with the initial condition (1.3).

Theorem 2.1. *Let $y_0(t)$ be a solution corresponding to $\mu_0 \in A$ defined on $[r_1, r_2] \subset (a, b)$ and let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $K_0 = \varphi_0(I_1) \cup y_0([r_1, r_2])$. Then the following conditions hold:*

2.1. *there exist numbers $\delta_i > 0$, $i = 0, 1$ such that a solution $y(t; \mu)$ defined on $[r_1 - \delta_1, r_2 + \delta_1] \subset I$ corresponds to each element*

$$\mu = (t_0, \tau, \varphi, f_0 + \delta f) \in V(\mu_0; K_1, \delta_0, \alpha).$$

Moreover,

$$\varphi(t) \in K_1, \quad t \in I_1; \qquad y(t; \mu) \in K_1, \quad t \in [r_1 - \delta_1, r_2 + \delta_1],$$

for arbitrary $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$;

2.2. *for an arbitrary $\varepsilon > 0$ there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ such that the following inequality holds for any $\mu \in V(\mu_0; K_1, \delta_2, \alpha)$*

$$|y(t; \mu) - y(t; \mu_0)| \le \varepsilon, \quad \forall t \in [r_1 - \delta_1, r_2 + \delta_1]. \tag{2.3}$$

Proof. Let $\varepsilon_0 > 0$ be so small that a closed ε_0-neighborhood of the set K_0:

$$K(\varepsilon_0) = \{x \in \mathbb{R}^n : \exists \hat{x} \in K_0 \ |x - \hat{x}| \le \varepsilon_0\}$$

lies in $int K_1$. There exist a compact set $Q : K_0^2(\varepsilon_0) \subset Q \subset K_1^2$ and a continuously differentiable function $\chi : \mathbb{R}^{2n} \to [0, 1]$ such that,

$$\chi(x_1, x_2) = \begin{cases} 1 & \text{for } (x_1, x_2) \in Q, \\ 0 & \text{for } (x_1, x_2) \notin K_1^2 \end{cases} \tag{2.4}$$

(see Assertion 3.2 in [1, p. 41]).

To each element $\mu \in A$, we assign the functional differential equation

$$\dot{z}(t) = \int_{\tau(t)}^{t} g(t, z(t), h(t_0, \varphi, z)(s))ds \tag{2.5}$$

with the initial condition

$$z(t_0) = \varphi(t_0), \tag{2.6}$$

where $g = \chi f$.

The function $g(t, x_1, x_2)$ satisfies the conditions:

$$|g(t, x_1, x_2)| \le m_{f,K_1}(t), \quad \forall x_i \in \mathbb{R}^n,\ i = 1, 2, \tag{2.7}$$

for $\forall x_i', x_i'' \in \mathbb{R}^n$, $i = 1, 2$ and for almost all $t \in I$

$$|g(t, x_1', x_2') - g(t, x_1'', x_2'')| \le L_f(t) \sum_{i=1}^{2} |x_i' - x_i''|, \tag{2.8}$$

where

$$L_f(t) = L_{f,K_1}(t) + \alpha_1 m_{f,K_1}(t), \qquad \alpha_1 = \sup\left\{\sum_{i=1}^{2} |\chi_{x_i}(x_1, x_2)| : x_i \in \mathbb{R}^n, \quad i = 1, 2\right\} \tag{2.9}$$

(see [8]).

It is clear that if $f = f_0 + \delta f$ then

$$L_{f,K_1}(t) = L_{f_0,K_1}(t) + L_{\delta f,K_1}(t), \qquad m_{f,K_1}(t) = m_{f_0,K_1}(t) + m_{\delta f,K_1}(t). \tag{2.10}$$

The solution of Eq. (2.5) with the initial condition (2.6) depends on the parameter

$$\mu = (t_0, \tau, \varphi, f_0 + \delta f) \in A_0 = [a, b) \times D \times \Phi \times \Big(f_0 + W(K_1; \alpha)\Big) \subset E_\mu.$$

The topology in A_0 is induced from the vector space E_μ.

On the complete metric space $C(I)$ with the distance $d(y_1, y_2) = \|y_1 - y_2\|_I$ we introduce a family

$$F(\cdot;\mu) : C(I) \to C(I) \tag{2.11}$$

of mapping depending on the parameter $\mu \in A_0$ by the formula

$$\zeta(t) = \zeta(t; z, \mu) = \varphi(t_0) + \int_{t_0}^{t}\left[\int_{\tau(\xi)}^{\xi} g(\xi, z(\xi), h(t_0, \varphi, z)(s))ds\right]d\xi,$$

where $g = \chi(f_0 + \delta f)$.

Clearly, every fixed point $z(t;\mu)$, $t \in I$, of mapping (2.11) is a solution of Eq. (2.5) with the initial condition (2.6).

Define the kth iteration $F^k(z;\mu)$ by

$$\zeta_k(t) = \zeta_k(t; z, \mu) = \varphi(t_0) + \int_{t_0}^{t}\left[\int_{\tau(\xi)}^{\xi} g(\xi, \zeta_{k-1}(\xi), h(t_0, \varphi, \zeta_{k-1})(s))ds\right]d\xi,$$
$$k = 1, 2, \ldots,\quad \zeta_0(t) = z(t).$$

Now let us prove that for a sufficiently large k, the family of mappings $F^k(z;\mu)$ is uniformly contractive. For this purpose, we estimate the difference

$$\begin{aligned}|\zeta_k'(t) - \zeta_k''(t)| &= |\zeta_k(t; z', \mu) - \zeta_k(t; z'', \mu)| \le \int_a^t\left[\int_{\tau(\xi)}^{\xi} |g(\xi, \zeta_{k-1}'(\xi), h(t_0, \varphi, \zeta_{k-1}')(s))\right.\\ &\quad \left.- g(\xi, \zeta_{k-1}''(\xi), h(t_0, \varphi, \zeta_{k-1}'')(s))|ds\right]d\xi \le \int_a^t\left[\int_{\tau(\xi)}^{\xi} L_f(\xi)\Big(|\zeta_{k-1}'(\xi) - \zeta_{k-1}''(\xi)|\right.\\ &\quad \left. + |h(t_0, \varphi, \zeta_{k-1}')(s) - h(t_0, \varphi, \zeta_{k-1}'')(s)|\Big)ds\right]d\xi, \quad i = 1, 2, \ldots\end{aligned} \tag{2.12}$$

(see (2.8)), where a function $L_f(\xi)$ has the form (2.9) i.e.

$$L_f(\xi) = L_{f_0+\delta f, K_1}(\xi) + \alpha_1 m_{f_0+\delta f, K_1}(\xi) = L_{f_0, K_1}(\xi) + L_{\delta f, K_1}(\xi) + \alpha_1[m_{f_0, K_1}(\xi) + m_{\delta f, K_1}(\xi)] \tag{2.13}$$

(see (2.10)).

Here, it is assumed that $\zeta_0' = z'(t)$ and $\zeta_0'' = z''(t)$.

It follows from the definition of the operator $h(\cdot)$ that

$$h(t_0, \varphi, \zeta_{k-1}')(s) - h(t_0, \varphi, \zeta_{k-1}'')(s) = h(t_0, 0, \zeta_{k-1}' - \zeta_{k-1}'')(s).$$

Using the last equality from relation (2.12) it follows that

$$\begin{aligned}|\zeta_k'(t) - \zeta_k''(t)| &\le 2\int_a^t L_f(\xi)(\xi - \tau(\xi)) \max_{\theta\in[a,\xi]} |\zeta_{k-1}'(\theta) - \zeta_{k-1}''(\theta)|d\xi\\ &\le 2(b - \tau(a))\int_a^t L_f(\xi) \max_{\theta\in[a,\xi]} |\zeta_{k-1}'(\theta) - \zeta_{k-1}''(\theta)|d\xi.\end{aligned}$$

Furthermore,

$$\max_{\theta\in[a,\xi]} |\zeta_{k-1}'(\theta) - \zeta_{k-1}''(\theta)| \le 2(b - \tau(a))\int_a^{\xi} L_f(\xi_1) \max_{\theta\in[a,\xi_1]} |\zeta_{k-2}'(\theta) - \zeta_{k-2}''(\theta)|d\xi_1.$$

Therefore

$$|\zeta_k'(t) - \zeta_k''(t)| \le [2(b - \tau(a))]^2\int_a^t L_f(\xi_1)d\xi_1\int_a^{\xi_1} L_f(\xi_2) \max_{\theta\in[a,\xi_2]} |\zeta_{k-2}'(\theta) - \zeta_{k-2}''(\theta)|d\xi_2.$$

By continuing this procedure, we obtain

$$|\zeta_k'(t) - \zeta_k''(t)| \le [2(b - \tau(a))]^k\alpha_k(t)\|z' - z''\|,$$

where

$$\alpha_k(t) = \int_a^t L_f(\xi_1)d\xi_1 \int_a^{\xi_1} L_f(\xi_2)d\xi_2 \dots \int_a^{\xi_{k-1}} L_f(\xi_k)d\xi_k.$$

By induction, one can readily show that

$$\alpha_k(t) = \frac{1}{k!}\left(\int_a^t L_f(\xi)d\xi\right)^k.$$

Thus,

$$d(F^k(z';\mu), F^k(z'';\mu)) = \|\zeta_k' - \zeta_k''\|_I \le \frac{[2(b-\tau(a))]^k}{k!}\left(\int_a^b L_f(\xi)d\xi\right)^k \|z'-z''\|_I$$
$$= \hat{\alpha}_k\|z'-z''\|_I.$$

Let us prove the existence of a number $\alpha_2 > 0$ such that

$$\int_I L_f(t)dt \le \alpha_2, \quad \forall f \in f_0 + W(K_1;\alpha).$$

Indeed, by (2.13) we have

$$\int_I L_f(t)dt = \int_I (L_{f,K_1}(t) + \alpha_1 m_{f,K_1}(t))dt = \int_I [L_{f_0,K_1}(t) + L_{\delta f,K_1}(t) + \alpha_1(m_{f_0,K_1}(t)$$
$$+ m_{\delta f,K_1}(t))]dt \le \alpha(\alpha_1+1) + \int_I [L_{f_0,K_1}(t) + \alpha_1 m_{f_0,K_1}(t)]dt = \alpha_2.$$

Taking into account this estimate, we obtain $\hat{\alpha}_k \le [2(b-\tau(a))\alpha_2]^k/k!$. Consequently, there exists a positive integer k_1 such that $\hat{\alpha}_{k_1} < 1$. Therefore, the k_1st iteration of the family (2.11) is contracting. By the fixed point theorem for contraction mappings (see [1, p. 61], [15, p. 110]), the mapping (2.11) has a unique fixed point for each μ. Hence it follows that Eq. (2.5) with the initial condition (2.6) has a unique solution $z(t;\mu)$, $t \in I$.

Let us prove that the mapping $F^k(z(\cdot;\mu_0);\cdot) : A_0 \to C(I)$ is continuous at the point $\mu = \mu_0$ for an arbitrary $k = 1, 2, \dots$. For this purpose, it suffices to show that if a sequence $\mu_i = (t_{0i}, \tau_i, \varphi_i, f_i) \in A_0$, $i = 1, 2, \dots$, where $f_i = f_0 + \delta f_i$, converges to $\mu_0 = (t_{00}, \tau_0, \varphi_0, f_0)$, i.e. if

$$\lim_{i\to\infty}\Big(|t_{0i} - t_{00}| + \|\tau_i - \tau_0\|_I + \|\varphi_i - \varphi_0\|_{I_1} + H(\delta f_i; K_1)\Big) = 0$$

then

$$\lim_{i\to\infty} F^k(z(\cdot;\mu_0);\mu_i) = F^k(z(\cdot;\mu_0);\mu_0) = z(\cdot;\mu_0). \tag{2.14}$$

We prove relation (2.14) by induction. Let $k = 1$, then we have

$$|\zeta_1^i(t) - z_0(t)| \le |\varphi_i(t_{0i}) - \varphi_0(t_{00})| + \left|\int_{t_{0i}}^t \left[\int_{\tau_i(\xi)}^{\xi} g_i(\xi, z_0(\xi), h(t_{0i}, \varphi_i, z_0)(s))ds\right]d\xi\right.$$
$$\left. - \int_{t_{00}}^t \left[\int_{\tau_0(\xi)}^{\xi} g_0(\xi, z_0(\xi), h(t_{00}, \varphi_0, z_0)(s))ds\right]d\xi\right| \le \alpha_1^i + \alpha_2^i(t), \tag{2.15}$$

where

$$\zeta_1^i(t) = \zeta_1(t; z_0, \mu_i), \qquad z_0(t) = z(t;\mu_0), \qquad g_i = \chi f_i = g_0 + \delta g_i, \quad g_0 = \chi f_0,\ \delta g_i = \chi \delta f_i;$$

$$\alpha_1^i = |\varphi_i(t_{0i}) - \varphi_0(t_{00})| + \left|\int_{t_{0i}}^t \left[\int_{\tau_i(\xi)}^{\tau_0(\xi)} |g_i(\xi, z_0(\xi), h(t_{0i}, \varphi_i, z_0)(s))|ds\right]d\xi\right|$$
$$+ \left|\int_{t_{00}}^{t_{0i}} \left[\int_{\tau_0(\xi)}^{\xi} |g_0(\xi, z_0(\xi), h(t_{00}, \varphi_0, z_0)(s))|ds\right]d\xi\right|,$$

$$\alpha_2^i(t) = \left| \int_{t_{0i}}^{t} \left[\int_{\tau_0(\xi)}^{\xi} |g_i(\xi, z_0(\xi), h(t_{0i}, \varphi_i, z_0)(s)) - g_0(\xi, z_0(\xi), h(t_{00}, \varphi_0, z_0)(s))| ds \right] d\xi \right|.$$

According to (2.7) and (2.9) we have

$$\alpha_1^i \le |\varphi_i(t_{0i}) - \varphi_0(t_{00})| + \left| \int_{t_{0i}}^{t} \Big[(\tau_0(\xi) - \tau_i(\xi)) m_{f_i,K_1}(\xi) \Big] d\xi \right|$$
$$+ \left| \int_{t_{00}}^{t_{0i}} \Big[(\xi - \tau_0(\xi)) m_{f_0,K_1}(\xi) \Big] d\xi \right| \le |\varphi_i(t_{0i}) - \varphi_0(t_{00})|$$
$$+ \|\tau_0 - \tau_i\|_I \left[\alpha + \int_I m_{f_0,K_1}(t) dt \right] + (b - \tau_0(a)) \left| \int_{t_{0i}}^{t_{00}} m_{f_0,K_1}(t) dt \right|,$$

therefore,

$$\lim_{i \to \infty} \alpha_1^i = 0. \tag{2.16}$$

After elementary transformation we obtain

$$\alpha_2^i(t) \le \left| \int_{t_{0i}}^{t} \left[\int_{\tau_0(\xi)}^{\xi} |g_0(\xi, z_0(\xi), h(t_{0i}, \varphi_i, z_0)(s)) - g_0(\xi, z_0(\xi), h(t_{00}, \varphi_0, z_0)(s))| ds \right] d\xi \right|$$
$$+ \left| \int_{t_{0i}}^{t} \left[\int_{\tau_0(\xi)}^{\xi} |\delta g_i(\xi, z_0(\xi), h(t_{0i}, \varphi_i, z_0)(s)) - \delta g_i(\xi, z_0(\xi), h(t_{00}, \varphi_0, z_0)(s))| ds \right] d\xi \right|$$
$$+ \left| \int_{t_{0i}}^{t} \left[\int_{\tau_0(\xi)}^{\xi} \delta g_i(\xi, z_0(\xi), h(t_{00}, \varphi_0, z_0)(s)) ds \right] d\xi \right| \le \alpha_3^i + \alpha_4^i + \alpha_5^i(t),$$

where

$$\alpha_3^i = \int_I L_{f_0}(\xi) \left[\int_{\tau_0(\xi)}^{\xi} |h(t_{0i}, \varphi_i, z_0)(s) - h(t_{00}, \varphi_0, z_0)(s)| ds \right] d\xi,$$
$$\alpha_4^i = \int_I L_{\delta f_i}(\xi) \left[\int_{\tau_0(\xi)}^{\xi} |h(t_{0i}, \varphi_i, z_0)(s) - h(t_{00}, \varphi_0, z_0)(s)| ds \right] d\xi,$$
$$\alpha_5^i(t) = \max_{t', t' \in I} \left| \int_{t'}^{t'} \left[\int_{\tau_0(\xi)}^{\xi} \delta g_i(\xi, z_0(\xi), h(t_{00}, \varphi_0, z_0)(s)) ds \right] d\xi \right|.$$

Introduce notation

$$s_{1i} = \min(t_{0i}, t_{00}), \qquad s_{2i} = \max(t_{0i}, t_{00}).$$

It is easy to see that

$$\lim_{i \to \infty} (s_{2i} - s_{1i}) = 0.$$

Now we estimate α_3^i and α_4^i. We have

$$\alpha_3^i \le \beta_i \int_I L_{f_0}(t) dt,$$

where

$$\beta_i = \|\varphi_i - \varphi_0\|_{I_1} (b - \tau(a)) + \int_{s_{1i}}^{s_{2i}} |\varphi_i(s) - z_0(s)| ds;$$
$$\alpha_4^i \le \beta_i \int_I L_{\delta f}(t) dt \le \alpha(1 + \alpha_1) \beta_i.$$

It is clear that $\beta_i \to 0$.

Thus,

$$\lim_{i\to\infty} \alpha_3^i = \lim_{i\to\infty} \alpha_4^i = 0. \tag{2.17}$$

Obviously,

$$H(\delta g_i; K_1) = H(\chi \delta f_i; K_1) \le H(\delta f_i; K_1)$$

(see (2.4)). Since $H(\delta f_i; K_1) \to 0$ as $i \to \infty$, we have

$$\lim_{i\to\infty} H(\delta g_i; K_1) = 0.$$

This allows us to use Lemma 2 given in [8], which in turn, implies

$$\lim_{i\to\infty} \alpha_5^i(t) = 0 \tag{2.18}$$

uniformly in $t \in I$.

Conditions (2.17) and (2.18) yield

$$\lim_{i\to\infty} \alpha_2^i(t) = 0 \tag{2.19}$$

uniformly in $t \in I$.

Taking into account (2.16) and (2.19) we get

$$\|\zeta_1^i - z_0\|_I = 0$$

(see (2.15)).

Relation (2.14) is proved for $k = 1$.

Let (2.14) hold for a certain $k > 1$; we will prove it for $k + 1$. Elementary transformations yield:

$$|\zeta_{k+1}^i(t) - z_0(t)| \le |\varphi_i(t_{0i}) - \varphi_0(t_{00})| + \left| \int_{t_{0i}}^t \left[\int_{\tau_i(\xi)}^{\xi} g_i(\xi, \zeta_k^i(\xi), h(t_{0i}, \varphi_i, \zeta_k^i)(s))ds \right] d\xi \right.$$
$$\left. - \int_{t_{00}}^t \left[\int_{\tau_0(\xi)}^{\xi} g_0(\xi, z_0(\xi), h(t_{00}, \varphi_0, z_0)(s))ds \right] d\xi \right| \le \alpha_1^i + \alpha_2^i(t) + \alpha_{3k}^i(t),$$

where

$$\alpha_{3k}^i(t) = \left| \int_{t_{0i}}^t \left[\int_{\tau_0(\xi)}^{\xi} |g_i(\xi, \zeta_k^i(\xi), h(t_{0i}, \varphi_i, \zeta_k^i)(s)) - g_i(\xi, z_0(\xi), h(t_{0i}, \varphi_i, z_0)(s))|ds \right] d\xi \right|.$$

The quantities α_1^i and $\alpha_2^i(t)$ have been estimated in the preceding, and it remains to estimate α_{3k}^i. We have

$$\alpha_{3k}^i \le \left| \int_{t_{0i}}^t L_{f_i}(\xi) \left[\int_{\tau_0(\xi)}^{\xi} \Big(|\zeta_k^i(\xi) - z_0(\xi)| + |h(t_{0i}, 0, \zeta_k^i - z_0)(s)| \Big) ds \right] d\xi \right.$$
$$\le \|\zeta_k^i - z_0\|_I (1 + b - \tau_0(a))\alpha_2.$$

Since

$$\lim_{i\to\infty} \|\zeta_k^i - z_0\|_I = 0$$

it follows that

$$\lim_{i\to\infty} \alpha_{4k}^i = 0. \tag{2.20}$$

According to (2.16), (2.19) and (2.20), we have

$$\lim_{i\to\infty} \|\zeta_{k+1}^i - z_0\|_I = 0.$$

Relation (2.14) is proved for every $k = 1, 2, \ldots$.

Let a number $\delta_1 > 0$ be so small that $[r_1 - \delta_1, r_2 + \delta_1] \subset I$ and $|z(t; \mu_0) - z(r_1; \mu_0)| \le \varepsilon_0/2$ for $t \in [r_1 - \delta_1, r_1]$ and $|z(t; \mu_0) - z(r_2; \mu_0)| \le \varepsilon_0/2$ for $t \in [r_2, r_2 + \delta_1]$.

We can conclude from the uniqueness of the solution $z(t; \mu_0)$ that $z(t; \mu_0) = y_0(t)$ for $t \in [r_1, r_2]$. Taking into account the above inequalities, we have

$$(z(t; \mu_0), h(t_{00}, \varphi_0, z(\cdot; \mu_0))(s)) \in K^2(\varepsilon_0/2) \subset Q, \quad t \in [r_1 - \delta_1, r_2 + \delta_1],\ s \in [\tau_0(t), t].$$

Hence

$$\chi(z(t; \mu_0), h(t_{00}, \varphi_0, z(\cdot; \mu_0))(s)) = 1, \quad t \in [r_1 - \delta_1, r_2 + \delta_1],\ s \in [\tau_0(t), t]$$

and the function $z(t; \mu_0)$ satisfies the equation

$$\dot y(t) = \int_{\tau_0(t)}^t f_0(t, y(t),\ h(t_0, \varphi, y)(s))ds, \quad t \in [r_1 - \delta_1, r_2 + \delta_1]$$

and the initial condition

$$y(t_{00}) = \varphi_0(t_{00}).$$

Therefore,

$$y(t; \mu_0) = z(t; \mu_0), \quad t \in [r_1 - \delta_1, r_2 + \delta_1].$$

According to the fixed point theorem, for $\varepsilon_0/2$ there exists a number $\delta_0 \in (0, \varepsilon_0)$ such that a solution $z(t; \mu)$ satisfying the condition

$$|z(t; \mu) - z(t; \mu_0)| \le \varepsilon_0/2, \quad t \in I,$$

corresponds to each element $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$. Therefore, for $t \in [r_1 - \delta_1, r_2 + \delta_1]$

$$z(t; \mu) \in K(\varepsilon_0) \forall \mu \in V(\mu_0; K_1, \delta_0, \alpha).$$

Taking into account that $\varphi(t) \in K(\varepsilon_0)$, we see that for $t \in [r_1 - \delta_1, r_2 + \delta_1]$ and $s \in [\tau(t), t]$ this implies

$$\chi(z(t; \mu), h(t_0, \varphi, z(\cdot; \mu))(s)) = 1 \quad \forall \mu \in V(\mu_0; K_1, \delta_0, \alpha).$$

Hence the function $z(t; \mu)$ satisfies Eq. (2.1) and condition (2.2), i.e.

$$y(t; \mu) = z(t; \mu) \in K_1, \quad t \in [r_1 - \delta_1, r_2 + \delta_1],\ \mu \in V(\mu_0; K_1, \delta_0, \alpha). \tag{2.21}$$

The first part of Theorem 2.1 is proved. By the fixed point theorem, for an arbitrary $\varepsilon > 0$, there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ such that for each $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$,

$$|z(t; \mu) - z(t; \mu_0)| \le \varepsilon, \quad t \in I.$$

Whence using (2.21), we obtain (2.3). □

Proof of Theorem 1.1. In Theorem 2.1, let $r_1 = t_{00}$ and $r_2 = t_{10}$. Obviously, the solution $x_0(t)$ satisfies the following equation on the interval $[t_{00}, t_{10}]$:

$$\dot y(t) = \int_{\tau_0(t)}^t f_0(t, y(t), h(t_{00}, \varphi, y)(s))ds.$$

Therefore, in Theorem 2.1, as the solution $y_0(t)$ we can take the function $x_0(t)$, $t \in [t_{00}, t_{10}]$.

By Theorem 2.1, there exist numbers $\delta_i > 0$, $i = 0, 1$, and for an arbitrary $\varepsilon > 0$, there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0]$ such that the solution $y(t; \mu)$, $t \in [t_{00} - \delta_1, t_{10} + \delta_1]$, corresponds to each $\mu \in V(\mu_0; K_1, \delta_2, \alpha)$. Moreover, the following conditions hold:

$$\begin{cases} \varphi(t) \in K_1, & t \in I_1;\ y(t; \mu) \in K_1, \\ |y(t : \mu) - y(t; \mu_0)| \le \varepsilon, & t \in [t_{00} - \delta_1, \quad t_{10} + \delta_1], \\ \mu \in V(\mu_0; K_1, \delta_2, \alpha). \end{cases} \tag{2.22}$$

For an arbitrary $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$, the function

$$x(t;\mu) = \begin{cases} \varphi(t), & t \in [\hat{\tau}, t_0), \\ y(t;\mu), & t \in [t_0, t_1 + \delta_1] \end{cases}$$

is the solution corresponding to μ. Moreover, if $t \in [\theta, t_{10} + \delta_1]$, then $x(t; \mu_0) = y(t; \mu_0)$ and $x(t; \mu) = y(t; \mu)$. Taking into account (2.22), we see that this implies 1.1 and 1.2. It is easy to note that for an arbitrary $\mu \in V(\mu_0; K_1, \delta_0, \alpha)$, we have

$$\begin{aligned} \int_{\hat{\tau}}^{t_{10}+\delta_1} |x(t;\mu) - x(t;\mu_0)|dt &= \int_{\hat{\tau}}^{\theta_0} |\varphi(t) - \varphi_0(t)|dt + \int_{\theta_0}^{\theta} |x(t;\mu) - x(t;\mu_0)|dt \\ &+ \int_{\theta}^{t_{10}+\delta_1} |x(t;\mu) - x(t;\mu_0)|dt \le \|\varphi - \varphi_0\|_{I_1}(b - \hat{\tau}) + M|t_0 - t_{00}| \\ &+ \max_{t\in[\theta, t_{10}+\delta_1]} |x(t;\mu) - x(t;\mu_0)|(b - \hat{\tau}), \end{aligned}$$

where $\theta_0 = \min\{t_0, t_{00}\}$, $M = \sup\{|x' - x''| : x', x'' \in K_1\}$.

By 1.1 and 1.2, this inequality implies 1.3. □

3. Proof of Theorem 1.3

To each element $\varrho \in A_1$ we will set in correspondence the functional differential equation

$$\dot{y}(t) = \int_{\tau(t)}^{t} g(t, y(t), h(t_0, \varphi, y))(s), u(t)ds, \tag{3.1}$$

with the initial condition (2.2).

Theorem 3.1. *Let $y_0(t)$ be a solution corresponding to $\varrho_0 = (t_{00}, \tau_0, \varphi_0, u_0) \in A_1$ defined on $[r_1, r_2] \subset (a, b)$ and let $K_1 \subset O$ be a compact set containing a certain neighborhood of the set $K_0 = \varphi_0(I_1) \cup y_0([r_1, r_2])$. Then the following conditions hold:*

3.1. *there exist numbers $\delta_i > 0$, $i = 0, 1$ such that to each element*

$$\varrho = (t_0, \tau, \varphi, u) \in \hat{V}(\varrho_0; \delta_0) = B(t_{00}; \delta_0) \times V(\tau_0; \delta_0) \times V_1(\varphi_0; \delta_0) \times V_2(u_0; \delta_0)$$

corresponds solution $y(t; \varrho)$ defined on the interval $[r_1 - \delta_1, r_2 + \delta_1] \subset I$ and satisfying the condition $y(t; \varrho) \in K_1$;

3.2. *for an arbitrary $\varepsilon > 0$, there exists a number $\delta_2 = \delta_2(\varepsilon) \in (0, \delta_0)$ such that the following inequality holds for any $\varrho \in \hat{V}(\varrho_0; \delta_2)$:*

$$|y(t;\rho) - y(t;\rho_0)| \le \varepsilon \quad \forall t \in [r_1 - \delta_1, r_2 + \delta_1].$$

Proof. Rewrite Eq. (3.1) in the form

$$\dot{y}(t) = \int_{\tau(t)}^{t} [g_0(t, y(t), h(t_0, \varphi, y)(s)) + \delta g_u(t, y(t), h(t_0, \varphi, y)(s))]ds,$$

where

$$g_0(t, x_1, x_2) = g(t, x_1, x_2, u_0(t)) \in E_f,$$
$$\delta g_u(t, x_1, x_2) = g(t, x_1, x_2, u(t)) - g_0(t, x_1, x_2) \in E_f.$$

Let $\hat{\delta}_0 > 0$ be a number so small that $V_2(u_0; \hat{\delta}_0) \subset \Omega$. There exists a compact set $M \subset U_0$ such that any function from the neighborhood $V_2(u_0; \hat{\delta}_0)$ assumes its values in M.

Let $K \subset O$ be a compact set. There exists a function $L_K(t) \in L(I, \mathbb{R}_+)$ such that for almost all $t \in I$, the following inequality holds:

$$|g(t, x_1', x_2'', u') - g(t, x_1', x_2'', u'')| \le L_K(t)\left[\sum_{i=1}^{2} |x_i' - x_i''| + |u' - u''|\right]$$
$$\forall x_i',\ x_i'' \in K,\ i = 1, 2,\ u', u'' \in M.$$

Hence

$$|\delta g_u(t,x_1,x_2)| \le L_K(t)|u(t)-u_0(t)| \le \hat{\delta}_0 L_K(t) \quad \forall x_i \in K, i=1,2,\ \forall u \in V_2(u_0;\hat{\delta}_0),$$

$$|\delta g_u(t,x_1',x_2') - \delta g_u(t,x_1'',x_2'')| \le 2L_K(t)\sum_{i=1}^{2}|x_i'-x_i''|, \quad \forall x_i', x_i'' \in K,\ i=1,2.$$

It is easy to see that the following inclusions hold for $\delta \in (0,\hat{\delta}_0]$:

$$\{\delta g_u(t,x_1,x_2) : u \in V_2(u_0;\delta)\} \subset W(K;\alpha),$$

$$\{\delta g_u(t,x_1,x_2) : u \in V_2(u_0;\delta)\} \subset V_{K,\hat{\delta}_1},$$

where

$$\alpha = (2+\hat{\delta}_0)\int_I L_K(t)dt, \qquad \hat{\delta}_1 = \delta\int_I L_K(t)dt.$$

Now we can use Theorem 2.1, which, is turn, proves Theorem 3.1. □

Proof of Theorem 1.3. In Theorem 3.1, let $r_1 = t_{00}$ and $r_2 = t_{10}$. Obviously, the solution $x_0(t)$ satisfies the following equation on the interval $[t_{00}, t_{10}]$:

$$\dot{y}(t) = \int_{\tau_0(t)}^{t} g(t, y(t), h(t_0,\varphi_0, y)(s), u_0(t))ds.$$

Therefore, in Theorem 3.1, as the solution $y_0(t)$, we can take the function $x_0(t)$, $t \in [t_{00}, t_{10}]$. After that, the proof of the theorem completely coincides with that of Theorem 1.1; for this purpose, it suffices to replace the element μ by the element ϱ and the set $V(\mu_0; K_1, \delta_0, \alpha)$ by the set $\hat{V}(\varrho_0;\delta_0)$ everywhere. □

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